1 The Existence and Uniqueness Theorem for First-Order Differential Equations

Let $I \subseteq \mathbb{R}$ be an open interval and $G \subseteq \mathbb{R}^n$, $n \geq 1$, be a domain.

**Definition 1.1** Let us consider a function $f : I \times G \rightarrow \mathbb{R}^n$. The general form of an explicit ordinary first-order differential equation is as follows:

$$\frac{dy}{dx} = f(x, y). \quad (1.1)$$

In this case, we shall say that the function $f$ defines a differential equation on $I \times G$. Also, $f$ will be often called the right hand-side of the differential equation (1.1). The variable $x$ is called the independent variable, while $y$ is the dependent variable or the unknown function.

**Proposition 1.2** Let $f : I \times G \rightarrow \mathbb{R}^n$ be a continuous function and let us consider the Cauchy problem:

$$\begin{cases} \frac{dy}{dx} = f(x, y), \\ y(x_0) = y_0, \quad (x_0, y_0) \in I \times G. \end{cases} \quad (1.2)$$

Also, let $I_0 \subseteq I$ be a neighbourhood of $x_0$. Then, the function $\varphi : I_0 \subseteq I \rightarrow G$ is a solution of the Cauchy problem (1.2) if and only if $\varphi$ is continuous and satisfies the relationship:

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(s, \varphi(s))ds, \quad \forall x \in I_0, \quad (1.3)$$

called the integral equation associated to problem (1.2).
Proof. Let us assume first that $\varphi : I_0 \subseteq I \to G$ is a solution of the Cauchy problem (1.2). Then, by definition, $\varphi$ is derivable (hence, continuous) and satisfies the equation, i.e.

$$\frac{d\varphi}{dx} = f(x, \varphi(x)), \quad \forall x \in I_0.$$ 

Therefore, $\varphi$ and the map $x \mapsto \int_{x_0}^{x} f(s, \varphi(s)) ds$ being primitives of the same function, there exists $C \in \mathbb{R}^n$ such that

$$\varphi(x) = C + \int_{x_0}^{x} f(s, \varphi(s)) ds.$$ 

But, since $\varphi(x_0) = y_0$, we get $C = y_0$ and we see that $\varphi(\cdot)$ satisfies the relationship (1.3).

Conversely, let us assume that $\varphi(\cdot)$ is continuous and satisfies (1.3). Then, obviously, $\varphi(\cdot)$ is derivable and

$$\begin{cases} 
\frac{d\varphi}{dx} = f(x, \varphi(x)), \\
\varphi(x_0) = y_0. 
\end{cases}$$

Hence, $\varphi(\cdot)$ is a solution of (1.2). \qed

Let us consider now the case $n = 1$. We will now see that rather mild conditions on the right hand side of a differential equations give us local existence and uniqueness of solutions.

For a given point $(x_0, y_0)$, let us consider the rectangle

$$D = \{(x, y) \in \mathbb{R}^2 \mid x_0 - a \leq x \leq x_0 + a, y_0 - b \leq y \leq y_0 + b\}. \quad (1.4)$$
Theorem 1.3 Let $f : D \rightarrow \mathbb{R}$ be a continuous function on $D$, which satisfies, in $D$, a Lipschitz condition with respect to its second argument, i.e. there exists $L > 0$ such that for any $(x, y_1), (x, y_2) \in D$ we have

$$| f(x, y_1) - f(x, y_2) | \leq L | y_1 - y_2 |. \quad (1.5)$$

Then, for the equation

$$\frac{dy}{dx} = f(x, y), \quad (1.6)$$

there exists a unique solution $y = y(x)$, defined for $x_0 - H \leq x \leq x_0 + H$, that satisfies the condition $y(x_0) = y_0$. Here,

$$H < \min \left( a, \frac{b}{M}, \frac{1}{L} \right),$$

where

$$M = \max_D | f(x, y) |.$$

As we shall see, the unique solution of the above problem can be determined as the limit of a uniformly convergent sequence of functions, called the sequence of successive approximations. This sequence is defined by the following recurrence formula:

$$\begin{cases}
y_0(x) = y_0, \\
y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t)) dt, \quad n \geq 1.
\end{cases} \quad (1.7)$$
Remark 1.4 Before giving the proof of this theorem, let us notice that this is a local existence theorem. In fact, one can prove the existence of the desired solution on the interval $x_0 - H \leq x \leq x_0 + H$, where $H = \min \left( a, \frac{b}{M} \right)$. 

Also, notice that instead of asking the Lipschitz condition (1.5) to be fulfilled, one may ask the existence and the boundedness, in the absolute value, in $D$, of the partial derivative $\frac{\partial f}{\partial y}(x, y)$, which is a cruder condition, but a more easily verifiable one. 

In order to prove Theorem 1.3, let us remember one of the most well-known theorems on fixed points, called the contraction-mapping principle.

**Theorem 1.5** Let $(M, d)$ be a complete metric space and let $A : M \to M$ be a contraction map, i.e. a map for which there exists $\alpha \in (0, 1)$ such that

$$d(A[y], A[z]) \leq \alpha d(y, z), \quad \forall y, z \in M.$$  

Then, $A$ has a unique fixed point in $M$, i.e. there exists $\overline{y} \in M$ such that $A[\overline{y}] = \overline{y}$. This point can be found by the method of successive approximations:

$$\overline{y} = \lim_{n \to \infty} y_n,$$  

where

$$y_n = A[y_{n-1}], \quad n \geq 1$$  

and $y_0$ is an arbitrary point in the space $M$. 

**Proof of Theorem 1.3.** To prove this existence and uniqueness theorem, we shall apply the contraction-mapping principle. Let us consider the space...
$C$ of all the continuous functions defined on the interval $x_0 - h \leq x \leq x_0 + h$ and having the graph in the rectangle $D$, where $h \leq \min(a, \frac{b}{M})$ will be chosen more precisely later on.

If we endow this space with the distance

$$d(y, z) = \max_{x_0 - h \leq x \leq x_0 + h} |y(x) - z(x)|,$$

then $(C, d)$ is a complete metric space, called the space of uniform convergence, since, in fact, convergence in the sense of this metric signifies uniform convergence.

Using Proposition 1.2, we can replace the Cauchy problem

$$\begin{cases}
\frac{dy}{dx} = f(x, y), \\
y(x_0) = y_0
\end{cases}$$

by the equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds.$$ (1.12)

Let us consider the operator

$$A[y] = y_0 + \int_{x_0}^{x} f(s, y(s))ds,$$ (1.13)

which associates with every continuous function $y$ defined on $x_0 - h \leq x \leq x_0 + h$ a continuous function $A[y]$, defined on the same interval and having the graph in $D$. Indeed,

$$| \int_{x_0}^{x} f(s, y(s))ds | \leq Mh \leq b.$$
We shall prove that this operator $A$ is a contraction map on the complete metric space $C$, endowed with the distance $d$ given by (1.11).

Indeed, let $y, z \in C$. Since

$$d(A[y], A[z]) = \max_{x_0 - h \leq x \leq x_0 + h} \left| \int_{x_0}^{x} (f(s, y) - f(x, z)) \, dx \right|,$$

using the Lipschitz inequality, we get

$$d(A[y], A[z]) \leq L \max_{x_0 - h \leq x \leq x_0 + h} \left| \int_{x_0}^{x} \left| y(x) - z(x) \right| \, dx \right| \leq$$

$$\leq Lh \max_{x_0 - h \leq x \leq x_0 + h} \left| y(x) - z(x) \right| = Lhd(y, z).$$

If we choose $h$ such that $0 < Lh \leq \alpha < 1$, then $A$ satisfies

$$d(A[y], A[z]) \leq \alpha d(y, z), \quad \alpha \in (0, 1).$$

This proves that $A$ is a contraction on $C$. Hence, from the contraction-mapping principle, we know that there exists a unique fixed point $\bar{y}$ of the operator $A$.

Noticing that the integral equation can be written as

$$A[y] = y,$$

the existence of a unique fixed point of the operator $A$ implies exactly the existence of a unique continuous solution of the equation (1.6). Also, let us notice that this solution can be found by the method of successive approximations.\[\]
Example  Let us consider the domain

\[ D = \{ (x, y) \in \mathbb{R}^2 \mid |x| \leq 1/2, |y - 1| \leq 1 \}. \]

On this domain, consider the problem

\[
\begin{cases}
\frac{dy}{dx} = xy, \\
y(0) = 1.
\end{cases}
\]  

(1.14)

Find the third-order approximation of the solution of problem (1.14).

It is not difficult to see that

\[ \frac{\partial f}{\partial y}(x, y) = x. \]

Therefore, on the domain \(D\), we obtain

\[ |\frac{\partial f}{\partial y}(x, y)| \leq \frac{1}{2} \]

and

\[ L = \max_{(x,y) \in D} |\frac{\partial f}{\partial y}(x, y)| = \frac{1}{2}. \]

Moreover,

\[ M = \max_{(x,y) \in D} |f(x, y)| = 1. \]

Hence, \(H < 1/2\) and the conditions of Theorem 1.3 are fulfilled on the interval \([-H, H]\). The required successive approximations are

\[ y_0(x) = 1, \]  

(1.15)
\[ y_1(x) = 1 + \int_0^x x\,dx = 1 + \frac{x^2}{2}, \quad (1.16) \]
\[ y_2(x) = 1 + \int_0^x (1 + \frac{x^2}{2})\,dx = 1 + \frac{x^2}{2} + \frac{x^4}{8}, \quad (1.17) \]
\[ y_3(x) = 1 + \int_0^x (1 + \frac{x^2}{2} + \frac{x^4}{8})\,dx = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48}. \quad (1.18) \]

If we stop at this step, the error we are introducing is
\[ \varepsilon_3(x) = | y(x) - y_3(x) | \leq ML^3 \frac{|x|^4}{4!}, \text{ for } |x| \leq 1/2. \]

Hence,
\[ \varepsilon_3(x) \leq \frac{|x|^4}{192}, \text{ for } |x| \leq 1/2. \]

So,
\[ \max_{|x| \leq 1/2} \varepsilon_3(x) \leq \frac{1}{3072}. \]

On the other hand, it is quite easy to get the exact solution of problem (1.14). We have
\[ y(x) = e^{x^2/2}. \]

Therefore, we notice that the third-order approximation \( y_3(x) \) contains the first four terms in Taylor’s expansion of the function \( e^{x^2/2} \) about the point \( x_0 = 0 \).

**Remark 1.6** (G. Peano, 1858-1932) The local existence of a solution of the equation (1.6) can be proven by a different method if we assume only the continuity of the function \( f \). However, this assumption is not enough to ensure the uniqueness of the solution.
Remark 1.7 Let us suppose that the right-hand side of the equation (1.6) depends also on a parameter $\mu$:

$$\frac{dy}{dx} = f(x, y, \mu). \tag{1.19}$$

If $f$ is continuous with respect to $\mu$ for $\mu_0 \leq \mu \leq \mu_1$, satisfies the conditions of the existence and uniqueness theorem (see Theorem 1.3) and if the Lipschitz constant $L$ is independent of $\mu$, then the solution $y = y(x, \mu)$ of the equation (1.19) that satisfies the initial condition $y(x_0) = y_0$ depends continuously on the parameter $\mu$.

Remark 1.8 Under similar conditions, it is possible to prove the continuous dependence of the solution $y = y(x, x_0, y_0)$ of the equation $\frac{dy}{dx} = f(x, y)$ on the initial values $x_0$ and $y_0$.

Remark 1.9 Let us consider the Cauchy problem

$$\begin{cases}
\frac{dy}{dx} = f(x, y), \\
y(x_0) = y_0.
\end{cases} \tag{1.20}$$

If in a neighbourhood of the point $(x_0, y_0)$ the function $f$ has continuous derivatives up to the order $k$, then, in some neighbourhood of the point $(x_0, y_0)$, the solution of problem (1.20) has continuous derivatives up to the order $(k + 1)$.

Let us consider again the differential equation

$$\frac{dy}{dx} = f(x, y). \tag{1.21}$$
If in the neighbourhood of the initial point \((x_0, y_0)\) the conditions of the existence and uniqueness theorem are fulfilled, then there is only one integral curve passing through this point.

If the conditions of the existence and uniqueness theorem are not fulfilled, various situations can appear. Through the point \((x_0, y_0)\) may pass one integral curve, several curves, an infinite number of integral curves or there is no integral curve that passes through this point.

The points at which at least one of the conditions of this uniqueness result are violated are called *singular points*.

A curve that consists entirely of singular points is called *singular*. If the graph of a certain solution consists entirely of singular points, then this solution is called *singular*.

Notice that not every point at which the conditions of the uniqueness result are violated is a singular one, since these conditions are only sufficient, but not necessary.

A solution of the differential equation (1.21) is *singular* if the corresponding integral curve has the following property: through any of its points, there passes another integral curve of the given equation which is tangent to the first curve.

In fact, the graph of a singular solution is just the envelope of the family of curves representing the general solution.

More precisely, let us suppose that we know the general solution

\[
\Phi(x, y, C) = 0
\]  

(1.22)
of the equation (1.21). Eliminating $C$ from this equation and from the equation
\[ \frac{\partial \Phi}{\partial C}(x, y, C) = 0, \]
we get
\[ \phi(x, y) = 0. \]
If this function satisfies our original differential equation, but it doesn’t belong to the family (1.22), this function will be the so-called singular solution. The uniqueness condition is violated at each point of such a singular integral and this singular solution consists only of singular points.

**Example 1.**

If we consider the differential equation
\[ \frac{dy}{dx} = \sqrt{1 - y^2}, \]
it is not difficult to see that we get the general solution
\[ (x + C)^2 + y^2 = 1, \quad C \in \mathbb{R} \]
and the singular solutions $y = \pm 1$. Therefore, we can see that the family of integral curves consists of circles of radius 1 centered on the $Ox$-axis and the envelope of this family of circles consists exactly of the straight lines $y = \pm 1$.

Let us note that the points on the boundary of the domain of existence of solution are also called *singular points*. Contrary, any point belonging to
the interior of the domain of existence such that through this point we have a single integral curve passing through is called an ordinary point.

Also, let us notice that singular solutions cannot be obtained from the general solution by giving particular admissible values to the constants.

**Example 2.**

If we consider the differential equation

$$\frac{dy}{dx} = \frac{2y}{x}, \quad x > 0,$$

it is not difficult to see that we get the general solution

$$y = Cx^2, \quad C \in \mathbb{R},$$

which is a family of parabolas and the singular solution $y = 0$. Therefore, we can see that the family of integral curves consists of parabolas and the envelope of this family of curves consists exactly of the straight line $y = 0$.

**Example 3.**

If we consider the differential equation

$$\frac{dy}{dx} = y^2 + x^2,$$

it is not difficult to see that the conditions of the existence and uniqueness theorem are fulfilled in the neighbourhood of any point and, as a consequence, we do not have singular solutions.
In quite a similar way it is possible to deal with the case \( n > 1 \), i.e. to prove the theorem of existence and uniqueness of solution for the system of equations

\[
\begin{aligned}
\frac{dy_i}{dx} &= f_i(x, y_1, y_2, \ldots, y_n), \\
y_i(x_0) &= y_{i0}, \quad i = 1, 2, \ldots, n.
\end{aligned}
\]  

(1.23)

First, let us notice that we can replace this Cauchy problem by the equivalent integral equations

\[
y_i(x) = y_{i0} + \int_{x_0}^{x} f_i(x, y_1, y_2, \ldots, y_n) \, dx, \quad i = 1, 2, \ldots, n.
\]  

(1.24)

Now, let us consider the rectangle

\[ D = \left\{ (x, y_1, y_2, \ldots, y_n) \in \mathbb{R}^{n+1} \mid x_0 - a \leq x \leq x_0 + a, \ y_{i0} - b_i \leq y_i \leq y_{i0} + b_i, \right. \]

\[ \left. \quad i = 1, 2, \ldots, n. \right\} \]

**Theorem 1.10** Suppose that all the functions \( f_i(x, y_1, y_2, \ldots, y_n) \) are continuous on \( D \) and satisfy, in \( D \), a Lipschitz condition with respect to all the arguments starting with the second one, i.e. there exists \( L > 0 \) such that for any \((x, y_1, y_2, \ldots, y_n)\) and \((x, z_1, z_2, \ldots, z_n) \in D \) we have

\[
| f(x, y_1, y_2, \ldots, y_n) - f(x, z_1, z_2, \ldots, z_n) | \leq L \sum_{i=1}^{n} | y_i - z_i | .
\]  

(1.25)

Then, for the system

\[
\begin{aligned}
\frac{dy_i}{dx} &= f_i(x, y_1, y_2, \ldots, y_n), \\
y_i(x_0) &= y_{i0}, \quad i = 1, 2, \ldots, n
\end{aligned}
\]  

(1.26)
there exists a unique solution $Y(x) = (y_1(x), y_2(x), ..., y_n(x))$, defined for $x_0 - H \leq x \leq x_0 + H$, that satisfies the initial condition $Y(x_0) = Y_0$, with $Y_0 = (y_{10}, y_{20}, ..., y_{n0})$. Here, $H < \min \left( a, \frac{b_1}{M}, \frac{b_2}{M}, ..., \frac{b_n}{M} \frac{1}{L} \right)$, where

$$M = \max_{D} \left| f_i(x, y) \right|.$$  

Proof. We shall not enter into the details of the proof of this theorem, because it follows exactly the same steps in the proof of Theorem 1.3. 

Remark 1.11 Let $f : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. We shall suppose that $f$ is continuous and there exist two continuous functions $a, b : I \rightarrow \mathbb{R}_{+}$ and a real number $r > 0$ such that

$$\| f(x, y) \| \leq a(x) \| y \| + b(x), \quad \forall (x, y) \in I \times \mathbb{R}^n, \quad \| y \| > r.$$  

Then, $f$ possesses the property of global existence of solutions, i.e. for any $(x_0, y_0) \in I \times \mathbb{R}^n$, there exists $\varphi : I \rightarrow \mathbb{R}^n$ solution of the Cauchy problem

$$\begin{cases}
\frac{dy}{dx} = f(x, y), \\
y(x_0) = y_0.
\end{cases}$$

Similar questions regarding the existence and uniqueness of solutions arise for first-order differential equations not solved for the derivative, i.e. equations in the implicit form:

$$F(x, y, y') = 0,$$

where $F : D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$, $F = F(u, v, w) \in C^1(D)$. 

14